

# Bound states of massive fermions in the Aharonov–Bohm-like fields

V.R. Khalilov\*

*Faculty of Physics, Moscow State University, 119991, Moscow, Russia*

Bound states of massive fermions in the Aharonov–Bohm-like fields have analytically been studied. The Hamiltonians with the Aharonov–Bohm-like potentials are essentially singular and, so, require specification of a one-parameter self-adjoint extension. We construct self-adjoint Dirac Hamiltonians with the Aharonov–Bohm (AB) potential in 2+1 dimensions that are specified by boundary conditions at the origin. It is of interest that for some range of extension parameter the AB potential can bind relativistic charged massive fermions. The bound-state energy is determined by the AB magnetic flux, depends upon fermion spin and extension parameter; it is periodical function of the magnetic flux. We also construct self-adjoint Hamiltonians for the so-called Aharonov–Casher (AC) problem, show that nonrelativistic neutral massive fermions can be bound by the Aharonov–Casher background, determine the range of extension parameter in which fermion bound states exist and find their energies as well as wave functions.

PACS numbers: 03.65.Ge, 73.22.Pr, 11.10.Kk

Keywords: Aharonov–Bohm potential; Aharonov–Casher problem; Singular Hamiltonian; Self-adjoint extensions; Boundary conditions; Fermion bound states

---

\* khalilov@phys.msu.ru

## I. INTRODUCTION

The quantum Aharonov–Bohm effect [1] is an important phenomenon analyzed in various physical situations in numerous works (see e.g., Ref. [2]). Considering an electron travels in a region with the magnetic flux restricted to a thin solenoid, the electron wave function may develop a quantum (geometric) phase, which describes the real behavior of the electrons propagation. Thus, the AB vector potential can produce observable effects because the relative (gauge invariant) phase of the electron wave function, correlated with a nonvanishing gauge vector potential in the domain where the magnetic field vanishes, depends on the magnetic flux [3].

It was observed that the Aharonov–Bohm problem is governed by Hamiltonians that are essentially singular and so require specification of a one-parameter self-adjoint extension in order for them to be treated as self-adjoint quantum-mechanical operators [4–7]. Self-adjoint Hamiltonians are specified by boundary conditions at the singular point.

One-parameter self-adjoint extensions of the Dirac Hamiltonian for the AB problem in 2+1 dimensions were constructed in [5, 6, 8]. In [5] a formal solution was constructed, which describes a bound fermion state in the field of cosmic string. New great interest to different effects in the two-dimensional systems has appeared recently after successful fabrication of graphene (see, [9–11]). We note while a description of electron states in the graphene in [12–14] were based on the Dirac equation for massless fermions, work [15] has shown that the massive case can also be created.

It seems that the physical reason for additional specification of the above Dirac Hamiltonians is also related to the interaction between the fermion spin magnetic moment and the source field [16]. Since the interaction potential is repulsive or attractive for different signs of spin projection this feature must be taken into account in the behavior of wave functions at the origin. The existence of weakly bound electron states, which can emerge due to the interaction between the electron spin magnetic moment and the AB magnetic field in 3+1 dimensions, was shown in [17].

Fermion bound states can emerge in the Aharonov–Casher problem [18] of the motion of a neutral fermion with an anomalous magnetic moment (AMM) in the electric field of an electrically charged conducting long straight thin thread oriented perpendicularly to the plane of fermion motion resulting from the interaction between the AMM of the moving fermion and the electric field [19]. Authors [19] argue that such kind of point interaction also appears in several Aharonov–Bohm-like problems [20–24].

In this paper, we analyze the AB problem taking into account the fermion spin term in the Dirac Hamiltonian. We find all self-adjoint Dirac Hamiltonians as well as their spectra in the Aharonov–Bohm potential in 2+1 dimensions using the so-called form asymmetry method developed in Refs. [25, 26]. In particular, expressions for the wave functions and bound state energies are obtained as functions of the magnetic flux, spin and extension parameters. By constructing self-adjoint Hamiltonians for the Aharonov–Casher problem we show that fermion bound states exist and find their energies as well as wave functions. We note that the AB and AC scattering problems were studied in [16, 27] using corresponding self-adjoint Hamiltonians.

We shall adopt the units where  $c = \hbar = 1$ .

## II. SELF-ADJOINT RADIAL DIRAC HAMILTONIANS IN AN AHARONOV–BOHM POTENTIAL IN 2+1 DIMENSIONS

In two spatial dimensions, the Dirac  $\gamma^\mu$ -matrix algebra is known to be represented in terms of the two-dimensional Pauli matrices  $\sigma_j$  and the parameter  $s = \pm 1$  can be introduced to label two types of fermions [28] and is applied to characterize two states of the fermion spin (spin “up” and “down”) [29, 30]. Then, the Dirac Hamiltonian for a fermion of the mass  $m$  and charge  $e = -e_0 < 0$  in an Aharonov–Bohm  $A_0 = 0$ ,  $A_r = 0$ ,  $A_\varphi = B/r$ ,  $r = \sqrt{x^2 + y^2}$ ,  $\varphi = \arctan(y/x)$  potential, is

$$H_D = \sigma_1 P_2 - s \sigma_2 P_1 + \sigma_3 m, \quad (1)$$

where  $P_\mu = -i\partial_\mu - eA_\mu$  is the generalized fermion momentum operator (a three-vector). The Hamiltonian (1) should be defined as a self-adjoint operator in the Hilbert space  $\mathfrak{H} = L^2(\mathbb{R}^2)$  of square-integrable two-spinors  $\Psi(\mathbf{r})$ ,  $\mathbf{r} = (x, y)$  with the scalar product

$$(\Psi_1, \Psi_2) = \int \Psi_1^\dagger(\mathbf{r}) \Psi_2(\mathbf{r}) d\mathbf{r}, \quad d\mathbf{r} = dx dy. \quad (2)$$

The total angular momentum  $J \equiv L_z + s\sigma_3/2$ , where  $L_z \equiv -i\partial/\partial\varphi$ , commutes with  $H_D$ , therefore, we can consider separately in each eigenspace of the operator  $J$  and the total Hilbert space is a direct orthogonal sum of subspaces of  $J$ .

In the real (three-dimensional) space, the quantity  $B$  characterizes the flux of the magnetic field  $\mathbf{H} = (0, 0, H) = \nabla \times \mathbf{A} = B\delta(x)\delta(y)$  through the surface of infinitely thin (of the radius  $R \rightarrow 0$ ) solenoid. Thus, there appears the interaction potential of the electron spin magnetic moment with the magnetic field in the form  $-seB\delta(r)/r$ , which is singular and must influence the behavior of solutions at the origin. The “spin” potential is invariant under the changes  $e \rightarrow -e, s \rightarrow -s$ , and it hence suffices to consider only the case  $e = -e_0 < 0$  and  $e_0B \equiv \mu > 0$ ,  $\mu$  is the magnetic flux  $\Phi$  in units of the elementary magnetic flux  $\Phi_0 \equiv 2\pi/e_0$ . Then, the potential is attractive for  $s = -1$  and repulsive for  $s = 1$ . For cosmic strings  $\Phi = e/Q$ , where  $Q$  is the Higgs charge [5–7].

Eigenfunctions of the Hamiltonian (1) are (see, [31])

$$\Psi(t, \mathbf{r}) = \frac{1}{\sqrt{2\pi r}} \begin{pmatrix} f_1(r) \\ f_2(r)e^{is\varphi} \end{pmatrix} \exp(-iEt + il\varphi), \quad (3)$$

where  $E$  is the fermion energy,  $l$  is an integer. The wave function  $\Psi$  is an eigenfunction of the operator  $J$  with eigenvalue  $j = l + s/2$  and

$$\check{h}F = EF, \quad F = \begin{pmatrix} f_1(r) \\ f_2(r) \end{pmatrix}, \quad (4)$$

where

$$\check{h} = is\sigma_2 \frac{d}{dr} + \sigma_1 \frac{l + \mu + s/2}{r} + \sigma_3 m, \quad \mu \equiv e_0 B \quad (5)$$

Thus, the problem is reduced to that for the radial Hamiltonian  $\check{h}$  in the Hilbert space of doublets  $F(r)$  square-integrable on the half-line.

As was shown in [30, 31] any doublets  $F(r), G(r)$  of the Hilbert space  $\mathfrak{H} = \mathfrak{L}^2(0, \infty)$  must satisfy

$$\lim_{r \rightarrow 0} G^\dagger(r) i\sigma_2 F(r) = 0. \quad (6)$$

Then, for  $\nu = |l + \mu + s/2| \neq n/2, n = 1, 2, \dots$  needed linear independent solutions of (4) are (see, [30])

$$U_1(r; E) = A(kr)^{1/2} \left( \frac{2m}{k} \right)^\nu \Gamma(1/2 + \nu) e^{-i\frac{\pi}{4}(1-s)} \begin{pmatrix} \sqrt{E+m} J_{\nu-s/2}(kr) \\ \sqrt{E-m} J_{\nu+s/2}(kr) \end{pmatrix} \quad (7)$$

and

$$U_2(r; E) = B(kr)^{1/2} \left( \frac{2m}{k} \right)^{-\nu} \Gamma(1/2 - \nu) e^{i\frac{\pi}{4}(1+s)} \begin{pmatrix} \sqrt{E+m} J_{-\nu+s/2}(kr) \\ -\sqrt{E-m} J_{-\nu-s/2}(kr) \end{pmatrix}, \quad (8)$$

with the asymptotic behavior at  $r \rightarrow 0$ :

$$U_1(r; E) = (mr)^\nu \left( \frac{1+s}{1-s} \right) + O(r^{\nu+1}), \quad r \rightarrow 0,$$

$$U_2(r; E) = (mr)^{-\nu} \left( \frac{1-s}{1+s} \right) + O(r^{-\nu+1}), \quad r \rightarrow 0,$$

where  $A, B$  are complex constants,  $k = \sqrt{E^2 - m^2}$ , and  $J_\mu(z)$  are the Bessel functions as well as

$$V_1(r; E) = U_1(r; E) + \frac{1}{4s\lambda} \omega(E) U_2(r; E), \quad (9)$$

where  $\omega(E) = \text{Wr}(U_1, V_1)$  is the Wronskian:

$$\omega(E) = \text{Wr}(U_1, V_1) = \frac{\Gamma(2\nu)\Gamma[-\nu + (1-s)/2]}{\Gamma(-2\nu)\Gamma[\nu + (1-s)/2]} \frac{(2\lambda)^{-2\nu}}{m^{-2\nu}} 4s\lambda \equiv \frac{\tilde{w}(E)}{\Gamma(-2\nu)}, \quad (10)$$

where  $\lambda = \sqrt{m^2 - E^2}$ . The doublet  $V_1$  also can be represented via the MacDonald functions:

$$V_1(r; E) = C(mr)^{1/2} \left( \frac{m}{\lambda} \right)^{\nu-1/2} \frac{2}{\Gamma(1/2 - \nu)} \begin{pmatrix} K_{\nu-s/2}(\lambda r) \\ sK_{\nu+s/2}(\lambda r) \end{pmatrix}, \quad (11)$$

where  $C$  is a complex constant. We note that

$$\nu(\pm l, s = 1, \mu) = \nu(\pm l + 1, s = -1, \mu). \quad (12)$$

Any doublet of the domain  $D(h)$  must satisfy

$$(F^\dagger(r) i \sigma_2 F(r))|_{r=0} = (\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = 0. \quad (13)$$

$D(h)$  is the space of absolutely continuous doublets  $F(r)$  regular at  $r = 0$  with  $hF(r)$  belonging to  $\mathfrak{L}^2(0, \infty)$ .

If  $\nu > 1/2$  there exist only solutions belonging to the continuous spectrum (7). If  $0 < \nu < 1/2$  equation (13) is not satisfied and its left-hand side

$$(\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = 4s\lambda(\bar{c}_1 c_2 - \bar{c}_2 c_1). \quad (14)$$

Therefore the adjoint operator  $h^*$  is not symmetric and we need to construct the nontrivial self-adjoint extensions of the initial symmetric operator  $h^0$ . By means of the linear transformation

$$c_{1,2} \rightarrow c_\pm = c_1 \pm i c_2 \quad (15)$$

equation (14) is reduced to the quadratic diagonal form

$$(\bar{f}_1 f_2 - \bar{f}_2 f_1)|_{r=0} = -i4s\lambda(|c_+|^2 - |c_-|^2) \quad (16)$$

with the inertia indices  $(1, 1)$ , which means that the deficiency indices of the symmetric operator  $h^0$  for  $0 < \nu < 1/2$  are  $(1, 1)$ . Equation (13) will be satisfied for any  $c_-$  related to  $c_+$  by

$$c_- = e^{i\theta} c_+, \quad 0 \leq \theta \leq 2\pi, \quad 0 \sim 2\pi. \quad (17)$$

The angle  $\theta$  parameterizes the self-adjoint extensions  $h_\theta$  of the symmetric operator  $h^0$ . These self-adjoint extensions are different for various  $\theta$  except for two equivalent cases  $\theta = 0$  and  $\theta = 2\pi$ . If we denote  $\xi = \tan(\theta/2)$ , then the relation (17) is equivalent to

$$c_2 = -\xi c_1, \quad -\infty \leq \xi = \tan \frac{\theta}{2} \leq +\infty, \quad -\infty \sim +\infty. \quad (18)$$

The values of  $\xi = \pm\infty$  are equivalent; they imply  $c_1 = 0$  so we can consider only  $\xi = \infty$ . Hence, in the range  $0 < \nu < 1/2$  there is one-parameter  $U(1)$ -family of the operators  $h_\theta \equiv h_\xi$  with the domain  $D_\xi$

$$h_\xi: \begin{cases} D_\xi = \begin{cases} F(r) : F(r) \text{ are absolutely continuous in } (0, \infty), F, hF \in \mathfrak{L}^2(0, \infty), \\ F(r) = C \left[ (mr)^\nu \begin{pmatrix} 1+s \\ 1-s \end{pmatrix} - \xi(mr)^{-\nu} \begin{pmatrix} 1-s \\ 1+s \end{pmatrix} \right], r \rightarrow 0, -\infty < \xi < +\infty, \\ F(r) = C(mr)^{-\nu} \begin{pmatrix} 1-s \\ 1+s \end{pmatrix} + O(r^{1/2}), r \rightarrow 0, \xi = \infty \end{cases} \\ h_\xi F = hF, \end{cases} \quad (19)$$

where  $C$  is a complex constant. Then

$$U_\xi(r; E) = U_1(r; E) - \xi U_2(r; E) \quad (20)$$

and

$$V_1(r; E) \equiv V_\xi = U_\xi(r; E) + \frac{1}{4s\lambda} \omega_\xi(E) U_2(r; E) \quad (21)$$

with

$$\omega_\xi(E) = \text{Wr}(U_\xi, V_\xi) = \omega(E) + 4s\lambda\xi, \quad (22)$$

where  $\omega(E)$  is determined by (10). For  $-\infty < \xi < \infty$ , the energy eigenstates (doublets) in the range  $|E| \geq m$  are

$$F(r) = U_1(r; E) - \xi U_2(r; E),$$

where  $U_1(r; E)$  and  $U_2(r; E)$  are determined by (7) and (8) with  $0 < \nu < 1$ . The operator  $h^0$  is not determined as an unique self-adjoint operator and so the additional specification of its domain, given with the real parameter  $\xi$  (the self-adjoint extension parameter) is required in terms of the self-adjoint

boundary conditions. It is well to note that the self-adjoint boundary conditions permit an integrable singularity in the wave functions at origin. Physically, they show that the probability current density is equal to zero at the origin.

The spectrum of the radial Hamiltonian is determined by (see [25, 31])

$$\frac{d\sigma(E)}{dE} = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \text{Im} \frac{1}{\omega_\xi(E + i\epsilon)}, \quad (23)$$

where the generalized function  $\omega_\xi(E + i\epsilon)$  is obtained by the analytic continuation of the corresponding Wronskian in the complex plane of  $E$ . It coincides with the corresponding function  $\omega(E)$  on the real axis of  $E$ . It can be verified that in the range  $|E| > m$  the functions  $\omega(E)$  and  $\omega_\xi(E)$  are continuous, complex-valued and not equal to zero for real  $E$ ; the spectral function  $\sigma(E)$  exists and is absolutely continuous. Thus, the energy spectrum in the range  $|E| \geq m$  is continuous. In the range  $|E| < m$  ( $-m < E < m$ ) the functions  $\omega(E)$  and  $\omega_\xi(E)$  are real and  $\lim_{\epsilon \rightarrow 0} \omega_\xi^{-1}(E + i\epsilon)$  can be complex only at the points where  $\omega_\xi(E) = 0$  and the energy spectrum of bound states is determined by roots of this equation. The Wronskians as a function of the complex  $E$  have two cuts  $(-\infty, -m]$  and  $[m, \infty)$  in the complex plane of  $E$ , so we determine the first (second) sheet with  $\text{Re}\lambda > 0$  ( $\text{Re}\lambda < 0$ ) on the real axis of  $E$ . Real bound states are situated on the first (physical) sheet.

### III. RELATIVISTIC BOUND FERMION STATES IN 2+1 DIMENSIONS

For negative  $\xi$  there exists a bound state. The bound-state energy  $E_\xi(\nu, s)$  is implicitly determined by equation  $\omega_\xi(E) = 0$ , i.e.

$$\frac{\Gamma(2\nu)\Gamma(-\nu + (1-s)/2)}{\Gamma(-2\nu)\Gamma(\nu + (1-s)/2)} \frac{(\lambda)^{-2\nu}}{m^{-2\nu}} = \xi. \quad (24)$$

Let us write

$$\mu = [\mu] + \beta \equiv n + \beta, \quad (25)$$

where  $[\mu] \equiv n$  denotes the largest integer  $\leq \mu$ , and  $1 > \beta \geq 0$ . Hence  $n = 0, 1, 2, \dots$  for  $\mu > 0$  and  $n = -1, -2, -3, \dots$  for  $\mu < 0$ . Since signs of  $e$  and  $B$  are fixed it is enough to consider the only case  $\mu > 0$ . One can suppose that a bound state exists due to the interaction of the fermion spin magnetic moment with AB magnetic field.

We define particle bound states as the states that tend to the boundary of the continuous spectrum  $E = m$  upon adiabatically slow switching of the external field (see, for instance [32, 33]). For  $l + n = 0, \mu = \beta > 0$  the only (particle) bound state  $s = -1$  satisfies self-adjoint condition (19). Rewrite (24) for this case as follows

$$\frac{\Gamma(1-2\beta)\Gamma(1/2+\beta)}{\Gamma(2\beta-1)\Gamma(3/2-\beta)} \left(\frac{m}{\lambda}\right)^{2\beta-1} = \xi, \quad 1/2 > \beta > 0 \quad (26)$$

and

$$\frac{\Gamma(2\beta-1)\Gamma(3/2-\beta)}{\Gamma(1-2\beta)\Gamma(1/2+\beta)} \left(\frac{m}{\lambda}\right)^{1-2\beta} = \xi, \quad 1 > \beta > 1/2. \quad (27)$$

It is easily to see that these equations keep for  $l + n = -1, s = 1$ . Since  $K_{-\gamma}(z) = K_\gamma(z)$  it is seen from Eq. (11) that bound fermion states with  $l + n = 0, s = -1$  or  $l + n = -1, s = 1$  are doublets represented via two MacDonald functions  $K_{1-\beta}(\lambda r)$  and  $K_\beta(\lambda r)$ .

It follows from Eqs. (26) and (27) that an adiabatic increase of the magnetic flux  $\mu$  between the integers  $n \rightarrow n + 1$  lifts an energy level  $E = m \rightarrow E = -m$  (see, also [5]) on the physical sheet  $\text{Re}\lambda > 0$  and  $E = -m \rightarrow E = m$  on the second (unphysical) sheet  $\text{Re}\lambda < 0$ . The second sheet is below the first one. The given bound-state energy is decreased (increased)  $E = m \rightarrow E = -m$  for  $\text{Re}\lambda > 0$  ( $E = -m \rightarrow E = m$  for  $\text{Re}\lambda < 0$ ) upon adiabatic increase of the flux  $\Phi$  between the integers  $n \rightarrow n + 1$  and is increased (decreased)  $E = -m \rightarrow E = m$  ( $E = m \rightarrow E = -m$ ) upon adiabatic increase of  $\Phi$  between  $n + 1 \rightarrow n + 2$ . Therefore, any bound-state energy is a periodic function of the magnetic flux similar to the case of the fermion motion in the Aharonov-Bohm potential along a closed circle [34]; it is repeated every time we change  $\mu$  by an integer. It is interesting that the induced current due to vacuum polarization in the AB field is finite periodical function of the magnetic flux [35].

For  $\xi = -1$  any curve  $E(\beta)$  is symmetric upon reflection with respect to the point  $\beta = 1/2, E = 0$ . One also can see there exists at  $\beta = 1/2$  a normalizable state with  $E = 0$ ; for  $\xi$  it lies in the middle of the gap  $2m$ . The wave function of this (particle) state is

$$F(r) = D(mr)^{1/2} \begin{pmatrix} 1 \\ s \end{pmatrix} K_{1/2}(mr). \quad (28)$$

We give few comments.

1. In the range of parameters  $0 > \xi > -\infty$  the constructed self-adjoint Hamiltonians  $h_\xi$  have real localized solutions (fermionic bound state); physically they exist if additional potential (in our case,  $s\mu\delta(\mathbf{r})$  type) is attractive.

2. We define antiparticle bound states as the states that tend to the boundary of the lower continuum  $E = -m$  upon adiabatically slow switching of the external field. Then, we can treat an antiparticle as a particle with opposite signs of  $e, s, E$  and we see that Dirac Hamiltonian (5) possesses a conjugation symmetry.

Jackiw and Rebbi [36] were observed that, in a time-inversion, charge conjugation symmetric theory of one-dimensional Dirac fermions interacting with a solitonic background field (the kink), the effective Hamiltonian possesses a conjugation symmetry. Because of this symmetry an isolated nondegenerate, charge-self-conjugate, zero-energy state (zero mode) lying in the middle of the gap  $2m$  exists [36–38] and the vacuum of the model must acquire a half-integer fermionic charge [36]. In the presence of a vector potential, the Dirac Hamiltonian does not exhibit a charge conjugation symmetry since a charge coupling treats particles and antiparticles differently. So the existence of fermion states with zero energy does not necessarily imply a fractional fermion number [39]. The presence of a magnetic field breaks time-inversion invariance.

In the considered case, the wave function (a doublet) of antiparticle  $F^a$  is related to that of particle  $F$  by means of the charge-conjugation operator given by the Pauli matrix  $C = \sigma_3$ , i.e. if  $F$  is a solution of the Dirac equation (4) with  $(l + \mu), s$  and energy  $E$ , then  $F^a = \sigma_3 F^*$  is also a solution of the same equation, but with  $-(l + \mu), -s, -E$ . For  $\xi = -1$  the antiparticle energy as a function of  $\beta$  is equal to zero at  $\beta = 1/2$ , and the wave function of antiparticle state with  $E^a = 0$  is  $F^a(r) = \sigma_3 F^*(r)$ , where  $F(r)$  is determined by (28). Therefore, the AB vector potential can yield bound states and localized spin-polarized charged zero modes (see, also [39, 40]). Since  $F^a(r)$  does not coincide with  $F(r)$  the fermionic charge keeps integer.

3. The behavior of the lowest particle energy level near the upper boundary  $E = -m$  of the lower continuum in the relativistic AB problem differs from the one in the cutoff Coulomb problem. In the (cut off) Coulomb problem, the lowest electron energy level can dive into the lower continuum  $[-m, -\infty)$ , then turn into resonance that can be described as a quasistationary state with “complex energy” (directly associated with the creation of electron-positron pair)[41](see, also, [42]); when the bound state pole disappears from the physical sheet the quasistationary state pole resides on the second (unphysical) sheet.

We see there are not particle bound states diving into the lower continuum, no quasistationary states with “complex energy” in the relativistic AB problem (there is not particle creation); also only fermionic bound states with real  $E$  can appear on the second sheet.

#### IV. BOUND FERMION STATES IN THE AHARONOV–CASHER PROBLEM

The Dirac–Pauli equation for a neutral fermion with the mass  $m$ , an AMM  $M$  in the form of the Schrödinger equation for the case of fermion motion in an electric field reads

$$i \frac{\partial \Psi}{\partial t} = H_{DP} \Psi \quad (29)$$

with the Hamiltonian

$$H_{DP} = \boldsymbol{\alpha} \cdot \mathbf{P} + iM\boldsymbol{\gamma} \cdot \mathbf{E} + \beta m. \quad (30)$$

Here  $\mathbf{P} = -i\nabla$  is the canonical momentum operator,  $\Psi$  is a bispinor,  $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ ,  $\boldsymbol{\alpha}$  are the Dirac matrices  $\mathbf{E}$  is the electric field strength.

Introducing the function

$$\Psi = \Psi_n e^{-imt} \quad (31)$$

and representing  $\Psi_n$  in the form

$$\Psi_n = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (32)$$

where  $\phi$  and  $\chi$  are spinors, we obtain an equation for the neutral fermion in the electric field of an electrically charged homogeneous long straight thin thread directed along the  $z$  axis in the nonrelativistic approximation in the form

$$i \frac{\partial \phi}{\partial t} = \frac{(\mathbf{P} - \mathbf{E} \times \mathbf{M})^2 - M^2 \mathbf{E}^2 + M \nabla \cdot \mathbf{E}}{2m} \phi, \quad (33)$$

where  $\mathbf{M} = M\boldsymbol{\sigma}$ ,  $\boldsymbol{\sigma}$  are the Pauli matrices and the term  $\nabla \cdot \mathbf{E}$  is equal to  $4\pi$  times the electric field charge density.

In the Aharonov–Casher field configuration

$$E_x = \frac{ax}{r^2}, \quad E_y = \frac{ay}{r^2}, \quad E_z = 0, \quad E_r = \frac{a}{r}, \quad E_\varphi = 0, \quad (34)$$

is the electric field for an electrically charged homogeneous long straight thin (a zero radius) thread and  $a/2$  is the total surface charge density. We also assume that the projection of the fermion momentum on the  $z$  axis is equal to zero. The radial component of the (macroscopic) electric field is determined by the mean surface charge density as  $\nabla \cdot \mathbf{E} = 4\pi\rho$ , and the expression  $\rho = a\delta(r)/4\pi r$ , therefore, well approximates  $\rho$ . We seek the solutions of (33) in the polar coordinates in the form

$$\phi(t, r, \varphi) = \exp(-iE_n t) \sum_{l=-\infty}^{\infty} F_l(r) \exp(il\varphi) \psi, \quad (35)$$

where  $E_n$  is the particle energy,  $l$  is an integer, and  $\psi$  is a constant spinor. The Hamiltonian of a neutral fermion in the Aharonov–Casher background contains only the matrix  $\sigma_3$ , and the wave function  $\phi$  therefore depends only on the number  $\zeta$  characterizing the conserved spin projection on the  $z$  axis, and its eigenvalue  $\zeta = \pm 1$  can be substituted for the operator  $\sigma_3$  in (33). After this substitution, the spin part of the wave function  $\psi$  becomes inessential, and we can consider only the scalar coordinate function  $\phi$  depending on  $\zeta$  (see, e.g., [43]). Thus, the radial Dirac–Pauli equation for the neutral fermion with AMM in the electric field of a thread oriented perpendicular to the plane of fermion motion in 3+1 dimensions in the nonrelativistic approximation coincides to the nonrelativistic equation in the Aharonov–Bohm problem and reads [16, 18]

$$h^n F_l(r) = E_n F_l(r), \quad h^n = -\frac{1}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{(l + Ma\zeta)^2}{r^2} - Ma \frac{\delta(r)}{r} \right). \quad (36)$$

Here  $E_n$  is related to  $E$  by  $E = m + E_n$ ,  $|E_n| \ll m$ . We also note that analogous singular term ( $\sim \delta(r)/r$ ) also appears in the quadratic Dirac equation in the AB problem; there it includes spin parameter in the form of an additional delta-function interaction of spin with magnetic field. The additional term must influence the behavior of solutions at the origin and it can be taken into account by means of boundary conditions at the point  $r = 0$ . In the nonrelativistic AC problem the boundary condition (13) can be given by [16] (see, also [44, 45])

$$(\bar{f}'f - \bar{f}f')|_{r=0} = 0, \quad (37)$$

where  $f(r) \equiv F_l(r)/\sqrt{r}$  and  $\bar{f}$  is the complex conjugate function  $f$ . Here we restrict ourselves with considering the case  $\gamma = |l + \zeta Ma| < 1$  when bound states can exist. Then, for each  $l$  in the range  $0 < \gamma < 1$  there is one-parameter  $U(1)$ -family of self-adjoint Hamiltonians  $h_\xi^n$  parameterized by (18) with the domain  $D_\xi^n$

$$h_\xi^n: \begin{cases} D_\xi^n = \begin{cases} f(r), f'(r) \text{ are absolutely continuous in } (0, \infty); f, h_\xi^n f \in \mathfrak{L}^2(0, \infty), \\ f(r) = A[(mr)^\gamma - \xi(mr)^{-\gamma}] + O(r), r \rightarrow 0, & -\infty < \xi < +\infty, \\ f(r) = A(mr)^{-\gamma}, r \rightarrow 0, & \xi = \infty \end{cases} \\ h_\xi^n f = h^n f, \end{cases} \quad (38)$$

where  $A$  is a complex constant. It is obvious that the function  $f(r)$  are the Bessel functions of the order  $\pm\gamma$ . Then, calculating the corresponding Wronskian we obtain

$$\omega(E_n) = \frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \left( \frac{2m}{\lambda} \right)^{2\gamma}, \quad (39)$$

where  $\lambda = \sqrt{-2mE_n}$ . By the analytic continuation of (39) in the complex plane of  $E_n$  we obtain the function  $\omega_\xi(E_n + i\epsilon)$ . Now the Wronskian as a function of the complex  $E_n$  have a cut  $(0, \infty)$  in the complex plane of  $E_n$  and the first (second) sheet is determined  $\text{Re}\sqrt{-2mE_n} > 0$  ( $\text{Re}\sqrt{-2mE_n} < 0$ ). Real bound states are situated on the first (physical) sheet.

It can be verified that in the range  $E_n > 0$  the functions  $\omega(E_n)$  and  $\omega_\xi(E_n)$  are continuous, complex-valued and not equal to zero for real  $E_n$ ; the function  $\sigma(E_n)$  exists and is absolutely continuous. Thus, the energy spectrum in this range is continuous. One can show there also exists a bound state (with  $E_n < 0$ ) in the range of parameters  $-\infty < \xi < 0$  for  $0 < \gamma < 1$  and its energy is determined by

$$\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \left( \sqrt{\frac{-E_n}{2m}} \right)^{-2\gamma} = -\xi. \quad (40)$$

The bound-state energy is the same on the first and second sheets; it is given by (compare with formula (90) in [19])

$$E_n = -2m \left( -\xi \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \right)^{-1/\gamma}. \quad (41)$$

The wave function of bound state is  $N\sqrt{mr}K_\gamma(\sqrt{-2mE_n}r)$  where  $N$  is a normalization factor. Since signs of  $M$  and  $a$  are fixed it is enough to consider the only (attractive) case  $Ma < 0$  and because of bound states exist for  $\gamma < 1$  we must have  $Ma < -1$ . It is seen there are bound states with  $\zeta = \pm 1$  for  $l = 0$  and with  $\zeta = 1(-1)$  for  $l = 1(-1)$ . Denote  $-Ma \equiv c > 0$  rewrite (41) for these cases as follows

$$E_n^0 = -2m \left( -\xi \frac{\Gamma(1-c)}{\Gamma(1+c)} \right)^{-1/c}, \quad l = 0, 0 < c < 1, \quad (42)$$

$$E_n^{\pm 1} = -2m \left( -\xi \frac{\Gamma(c)}{\Gamma(2-c)} \right)^{1/(c-1)}, \quad l = \pm 1, 0 < c < 1. \quad (43)$$

It is evident that  $E_n^0(c) = E_n^{\pm 1}(c = 1 - b)$ ,  $1 > b > 0$ . This means that an adiabatic increase of  $c$  in the interval  $(0, 1)$  lifts the levels  $E_n^0(c)$  on the first (physical) sheet and  $E_n^{\pm 1}(c)$  on the second (unphysical) sheet in the opposite direction. The second sheet is below the first one.

Special case  $\gamma = 0$  can be of some interest (analogous case was considered in [17, 44] for the non-relativistic AB problem in 2+1 dimensions). One can show that for  $|\xi| = \infty$  the energy spectrum is continuous and nonnegative as well as for  $-\infty < \xi < 0$  there exists (in addition to the continuous part of the spectrum) one negative level

$$E_0 = -4me^{2(\xi-C)}, \quad (44)$$

where  $C = 0.57721$  is the Euler constant [46]. The wave function of bound state for  $\gamma = 0$  is  $N\sqrt{mr}K_0(\sqrt{-2mE_0}r)$ .

## V. SUMMARY

By constructing a one-parameter self-adjoint extension of the Dirac Hamiltonian with the AB potential in 2+1 dimensions, we have studied bound states of fermions in this background. It has been shown that for negative values of extension parameter  $\xi$ , the spectrum of self-adjoint Dirac Hamiltonians, in addition to its continuous part, has one bound level, therefore, the Aharonov–Bohm vector potential can bind relativistic charged massive fermions in 2+1 dimensions. The bound-state energy depends upon extension parameter and is periodical function of the AB magnetic flux. It is of interest that the AB vector potential can yield localized spin-polarized charged zero modes.

We also have studied the Aharonov–Casher problem in the context of the nonrelativistic limit of the Dirac–Pauli equation in 3+1 dimensions. We show that the AC background can bind nonrelativistic neutral massive fermions, determine the range of extension parameter in which fermion bound states exist and find their energies as well as wave functions.



- [2] M. Peshkin and A. Tonomura, *The Aharonov–Bohm Effect*, (Springer-Verlag, Berlin, 1989).
- [3] K. Huang, *Quarks, Leptons, and Gauge Fields* (World Scientific, Singapore, 1982).
- [4] I.V. Tyutin, *Electron Scattering by a Solenoid*, Preprint of P.N. Lebedev Institute, No 27 (1974), unpublished; e-print arXiv:quant-ph/0801.2167 v2.
- [5] P. de Sousa Gerbert, Phys. Rev., **D40**, 1346 (1989).
- [6] M.G. Alford, J. March-Pussel and F. Wilczek, Nucl.Phys., **B328**, 140 (1989).
- [7] M.G. Alford and F. Wilczek, Phys. Rev. Lett., **62**, 1071 (1989).
- [8] V.R. Khalilov, Theoretical and Mathematical Physics, **163**, 511 (2010).
- [9] K. S. Novoselov et al., Science, **306**, 666 (2004).
- [10] A.H. Castro Neto, F. Guinea, N.M. Peres, K.S. Novoselov, and A.K. Geim, Rev. Mod. Phys., **81**, 109 (2009).
- [11] V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea and A. H. Castro Neto, Rev. Mod. Phys. **84**, 1067 (2012).
- [12] K.S. Novoselov et al, Nature, **438**, 197 (2005).
- [13] Z. Jiang, Y. Zhang, H.L. Stormer, and P. Kim, Phys. Rev. Lett., **99**, 106802 (2007).
- [14] I.F. Herbut, Phys. Rev. Lett., **104**, 066404 (2010).
- [15] F. Guinea, M.I. Katsnelson and A.K. Geim, Nat. Phys., **6** 30 (2009).
- [16] V.R. Khalilov, I.V. Mamsurov, Theoretical and Mathematical Physics, **161**, 1503 (2009).
- [17] V.R. Khalilov, Mod. Phys. Lett., **A21** 1647 (2006).
- [18] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).
- [19] E.O. Silva, F.M. Andrade, C. Filgueiras, and H. Belich, Eur. Phys. J., **C73**(4) 2402 (2013).
- [20] F.M. Andrade, E.O. Silva, M. Pereira, Phys. Rev., **D85**(4), 041701(R) (2012).
- [21] F.M. Andrade, E.O. Silva, Phys. Lett., **B719**(4-5), 467 (2013).
- [22] E.O. Silva, F.M. Andrade, Europhys. Lett., **101**(5), 51005 (2013).
- [23] C. Filgueiras, E.O. Silva, F.M. Andrade, J. Math. Phys., **53**(12), 122106 (2012).
- [24] B. Allen, B.S. Kay, A.C. Ottewill, Phys. Rev., **D53**(12), 6829 (1996).
- [25] B.L. Voronov, D.M. Gitman, and I.V. Tyutin, Theoretical and Mathematical Physics, **150**, 34 (2007).
- [26] D.M. Gitman, I.V. Tyutin, and B.L. Voronov, *Self-adjoint Extensions in Quantum Mechanics* (Springer Science+Business Media, New York, 2012).
- [27] V.R. Khalilov, K.-E. Lee, and I.V. Mamsurov, Mod. Phys. Lett., **A27**, No 5, 1250027 (2012).
- [28] Y. Hosotani, Phys. Lett., **B319**, 332 (1993).
- [29] C.R. Hagen, Phys. Rev. Lett., **64**, 503 (1990).
- [30] V.R. Khalilov and K.-E. Lee, Journ. Phys., **A44**, 205303 (2011).
- [31] V.R. Khalilov and K.-E. Lee, Mod. Phys. Lett., **A26**, No 12, 865 (2011).
- [32] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Quantum Electrodynamics*, 2nd edn. (Pergamon, New York, 1982).
- [33] V.N. Rodionov, I.M. Ternov, and V.R. Khalilov, ZhETF, **71**, No 9, 871 (1976).
- [34] D. Cohen, *Lecture Notes in Quantum Mechanics*, e-print arXiv:quant-ph/0605180v5 (2013).
- [35] R. Jackiw, A.I. Milstein, S.-Y. Pi, and I.S. Terekhov, *Induced Current and Aharonov–Bohm Effect in Graphene*, e-print arXiv:cond-mat.mes-hall/0904.2046v3, (2009).
- [36] R. Jackiw, and C. Rebbi, Phys. Rev., **D13**, 3398 (1976).
- [37] R. Jackiw, and S.-Y. Pi, Phys. Rev. Lett., **98**, 266402 (2007).
- [38] R. Jackiw, *FRACTIONAL AND MAJORANA FERMIONS: The Physics of Zero Energy Modes*, e-print arXiv:cond-mat.str-el/1104.4486v1, (2011).
- [39] C.-L. Ho, and V.R. Khalilov, Phys. Rev., **D63**, 027701 (2000).
- [40] J.M. Fonseca, W.A. Moura-Melo, and A.R. Pereira, *Bound-states and polarized charged zero modes in three-dimensional topological insulators induced by a magnetic vortex*, e-print arXiv:cond-mat.mes-hall/1210.3100v2, (2013).
- [41] W. Greiner, J. Reinhardt, *Quantum Electrodynamics*, 4<sup>th</sup> ed. (Springer-Verlag, Berlin Heidelberg, 2009).
- [42] V.R. Khalilov, Eur. Phys. J., **C73**(8) 21 (2013).
- [43] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, New York, 1977).
- [44] D.M. Gitman, A. Smirnov, I.V. Tyutin, and B.L. Voronov, *Self-adjoint Schrödinger and Dirac operators with Aharonov–Bohm and magnetic-solenoid fields*, e-print arXiv:quant-ph/0911.0946v1 (2009).
- [45] V.R. Khalilov, K.-E. Lee, and I.V. Mamsurov, *Free and bound spin-polarized fermions in the fields of Aharonov–Bohm kind*, e-print arXiv:quant-ph/1002.2826v1 (2010).
- [46] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5<sup>th</sup> ed. (Academic Press, San Diego, 1994).